Hilbert modular forms from orthogonal modular forms on quaternary lattices

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Lattices and quadratic forms

Let F be a totally real number field with ring of integers $R = \mathbb{Z}_F$.
Let $Q : V \to F$ be a totally positive definite quaternary ($\dim_F V = 4$) quadratic space with associated bilinear form

$$T(x, y) := Q(x + y) - Q(x) - Q(y).$$

Let $\Lambda \subseteq V$ be an even **integral** lattice, so that $Q(\Lambda) \subseteq R$.
Define $\operatorname{disc}(\Lambda) = \langle \det[T]_B : B \subseteq \Lambda \rangle \subseteq R$.
When $F = \mathbb{Q}$, we write $\operatorname{disc}(\Lambda) = D\mathbb{Z}$.

Theorem (Hecke (1940))

If $F = \mathbb{Q}$, N prime, and $D = N^2$, then

$$\theta_\Lambda(z) = \theta_{\Lambda,1}(z) = \sum_{\lambda \in \Lambda} q^{Q(\lambda)} \in M_2(N), \quad q = e^{2\pi i z}$$

Quaternion algebras

Let B be definite quaternion algebra over F, $\mathcal{O}$ an R-order in B.
Two right $\mathcal{O}$-ideals I, J are **isomorphic**, written $I \simeq_r J$, if there exists $\alpha \in B^\times$ such that $I = \alpha J$.
Let

$$\mathrm{Idl}_r(\mathcal{O}) = \{I \subseteq B : I_\mathfrak{p} \simeq_r \mathcal{O}_\mathfrak{p} \text{ for all } \mathfrak{p}\}$$

be the set of locally principal right $\mathcal{O}$-ideals.
The **(right) class set** $\mathrm{cls}(\mathcal{O}) = \mathrm{Idl}_r(\mathcal{O})/\simeq$ is the set of (global) isomorphism classes in $\mathrm{Idl}_r(\mathcal{O})$.
Then $\mathrm{nrd} : B \to F$ is a totally positive definite quadratic space, and for every $I \in \mathrm{Idl}_r(\mathcal{O})$, $\frac{1}{\mathrm{nrd}(I)} I$ is an even integral lattice.

Conjecture (Hecke (1940))

If N is prime, $\mathrm{disc}(B) = N$, $\mathcal{O}$ maximal order, then

$$\{\theta_{\Lambda_1} - \theta_{\Lambda_2} : \Lambda_1, \Lambda_2 \in \mathrm{cls}(\mathcal{O})\}$$

generate $S_2(N)$.

Eichler's Basis Problem

Example (Eichler (1955))

When $N = 37$, $\mathrm{cls}(\mathcal{O}) = \{[I_1], [I_2], [I_3]\}$, with $\theta_{I_2} = \theta_{I_3}$, while $\dim S_2(37) = 2$. Hecke's conjecture is false.

Theorem (Eichler (1955))

For prime N, there exist lattices $\{\Lambda_i\}$ of discriminant N^2 such that $\{\theta_{\Lambda_i} - \theta_{\Lambda_j}\}$ generate $S_2(N)$.

Theorem (Hijikata, Pizer, and Shemanske (1989))

For all N, there exist lattices $\{\Lambda_i\}$ of discriminant N^2 such that $\{\theta_{\Lambda_i} - \theta_{\Lambda_j}\}$ and their twists generate $S_2(N, \chi)$.

Genus and Class set

We define the orthogonal group

$$\mathrm{O}(V) = \{g \in \mathrm{GL}(V) : Q(gv) = Q(v)\}$$
$$\mathrm{O}(\Lambda) = \{g \in \mathrm{O}(V) : g\Lambda = \Lambda\}$$

and write $\mathrm{SO}(V)$ and $\mathrm{SO}(\Lambda)$ for those with $\det(g) = 1$.
Lattices Λ, Π are **isometric**, written $\Pi \simeq \Lambda$, if there exists $g \in \mathrm{O}(V)$ such that $g\Lambda = \Pi$.
The **genus** of $\Lambda \subseteq V$ is

$$\mathrm{gen}(\Lambda) := \{\Pi \subseteq V : \Lambda_{\mathfrak{p}} \simeq \Pi_{\mathfrak{p}} \text{ for all } \mathfrak{p}\}.$$

The **class set** $\mathrm{cls}(\Lambda) = \mathrm{gen}(\Lambda)/\simeq$ is the set of (global) isometry classes in $\mathrm{gen}(\Lambda)$.

Orthogonal point of view

Theorem (Eichler (1955))

If N is prime, $D = N^2$, then

$$\{\theta_{\Lambda_1} - \theta_{\Lambda_2} : \Lambda_1, \Lambda_2 \in \mathrm{cls}(\Lambda)\}$$

generate $S_2(N)$.

- What happens for $D \neq \square$?
- θ is not injective. Can we get modular forms without θ?

Fun with L-functions

Lattice Λ **magic** $\rightsquigarrow$ **orthogonal modular forms** ϕ_i.

Example ($n = 4$, $D = 37^2$)

For Λ with Gram matrix $\begin{pmatrix} 2 & 0 & 1 & 1 \\ 0 & 4 & 1 & 2 \\ 1 & 1 & 10 & 1 \\ 1 & 2 & 1 & 20 \end{pmatrix}$ we get

$$L_p(\phi_1, T) = (1 - T)(1 - pT)^2(1 - p^2 T)$$
$$L_p(\phi_2, T) = (1 - (a_p^2 - 2p)T + p^2 T^2)(1 - pT)^2$$
$$L_p(\phi_3, T) = (1 - (b_p^2 - 2p)T + p^2 T^2)(1 - pT)^2$$
$$L_p(\phi_4, T) = (1 - pb_p T + p^3 T^2)(1 - b_p T + pT^2)$$

where a_p, b_p are coefficients of 37.2.a.a and 37.2.a.b.

Symmetric Square L-functions

Lattice Λ **magic** $\rightsquigarrow$ **orthogonal modular forms** ϕ_i.

Example ($n = 4$, $D = 37^2$)

For Λ with Gram matrix $\begin{pmatrix} 2 & 0 & 1 & 1 \\ 0 & 4 & 1 & 2 \\ 1 & 1 & 10 & 1 \\ 1 & 2 & 1 & 20 \end{pmatrix}$ we get

$$L_p(\phi_1, T) = (1 - T)(1 - pT)^2(1 - p^2 T)$$
$$L_p(\phi_2, T) = (1 - pT)L_p(\mathrm{Sym}^2(f), T)$$
$$L_p(\phi_3, T) = (1 - pT)L_p(\mathrm{Sym}^2(g), T)$$
$$L_p(\phi_4, T) = L_p(E \otimes f, T)$$

where $f, g \in S_2(37)$ are 37.2.a.a and 37.2.a.b.

time ($p < 100$): 109.15s (2.3GHz 8-Core Intel Core i9)

Neighbors

Kneser's theory of $\mathfrak{p}^k$-neighbors gives an effective method to compute the class set.
Let $\mathfrak{p} \nmid \mathrm{disc}(\Lambda)$ be a prime; $\mathfrak{p} \mid 2$ is OK.
We say that an integral lattice $\Pi \subseteq V$ is a **$\mathfrak{p}^k$-neighbor** of Λ, and write $\Pi \sim_{\mathfrak{p}^k} \Lambda$ if

$$\Lambda/(\Lambda \cap \Pi) \simeq (R/\mathfrak{p}R)^k \simeq \Pi/(\Lambda \cap \Pi),$$

If $\Lambda \sim_{\mathfrak{p}^k} \Pi$ then $\Pi \in \mathrm{gen}(\Lambda)$.
Moreover, there exists S such that every $[\Pi] \in \mathrm{cls}(\Lambda)$ is an **iterated S-neighbor** of Λ.

$$\Lambda \sim_{\mathfrak{p}_1} \Lambda_1 \sim_{\mathfrak{p}_2} \cdots \sim_{\mathfrak{p}_r} \Lambda_r \simeq \Pi$$

with $\mathfrak{p}_i \in S$. Typically may take $S = \{\mathfrak{p}\}$.

Example - Computing the class set

Let $\Lambda = \mathbb{Z}^4$ with the quadratic form

$$Q(x_1, x_2, x_3, x_4) = x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_1x_4 + x_3x_4 + 3x_4^2$$

and bilinear form given by Let

$$\Lambda = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 1 & 6 \end{pmatrix}$$

Thus $\operatorname{disc}(\Lambda) = 29$. We have $\# \operatorname{cls}(\Lambda) = 2$, with the nontrivial class represented by the 2-neighbor

$$\Lambda' = \frac{1}{2}\mathbb{Z}(e_2 + e_4) + 2\mathbb{Z}e_3 + \mathbb{Z}e_1 + \mathbb{Z}e_4.$$

with corresponding quadratic form

$$Q(x) = x_1^2 + x_1x_2 + 4x_2^2 + x_1x_3 + x_3^2 + 3x_1x_4 + 2x_2x_4 + x_3x_4 + 3x_4^2$$

Orthogonal modular forms

The space of **orthogonal modular forms** of level Λ (and trivial weight) is

$$M(\Lambda) := \{\phi : \mathrm{cls}(\Lambda) \to \mathbb{Q}\} \simeq \mathbb{Q}^{h(\Lambda)}$$

For $\mathfrak{p} \nmid \mathrm{disc}(\Lambda)$ define the **Hecke operator**

$$\begin{aligned} T_{\mathfrak{p}^k} : M(\Lambda) &\to M(\Lambda) \\ \phi &\mapsto \left([\Lambda'] \mapsto \sum_{\Pi' \sim_{\mathfrak{p}^k} \Lambda'} \phi([\Pi']) \right) \end{aligned}$$

The Hecke operators commute and are self-adjoint, hence there is a basis of simultaneous eigenvectors - **eigenforms**. (Gross, 1999)

Example - square discriminant

Let Λ have the Gram matrix

$$[T_\Lambda] = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 6 & 0 \\ 1 & 0 & 0 & 6 \end{pmatrix}$$

so that $\mathrm{disc}(\Lambda) = \det T = 11^2$. Then $h(\Lambda) = 3$.
Write $\mathrm{cls}(\Lambda) = \{[\Lambda] = [\Lambda_1], [\Lambda_2], [\Lambda_3]\}$. Then a basis of eigenforms is given by

$$\phi_1 = [\Lambda_1] + [\Lambda_2] + [\Lambda_3], \qquad \phi_2 = 4[\Lambda_1] - 6[\Lambda_2] + 9[\Lambda_3]$$
$$\phi_3 = 4[\Lambda_1] + [\Lambda_2] - 6[\Lambda_3],$$

and we have

$$\theta(\phi_1) = \frac{5}{12} + q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + 12q^6 + O(q^7) \in E_2(11)$$
$$\theta(\phi_2) = q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 + O(q^9) \in S_2(11)$$

where $T_p(\phi_2) = \lambda_p \phi_2$ with $\lambda_2 = 4, \lambda_3 = 1, \lambda_5 = 1, \lambda_7 = 4, \ldots$

Back to L-functions

Letting $D^* = (-1)^{\frac{n}{2}} D$ there is a natural family of theta maps:

$$\theta^{(g)} : M(\Lambda) \to M_{\frac{n}{2}}(\Gamma_0^{(g)}(D), \chi_{D^*}).$$

Theorem (A., Fretwell, Ingalls, Logan, Secord, and Voight (2022), consequence of Rallis (1982))

If n is even, ϕ is an eigenform and $f = \theta^{(g)}(\phi) \neq 0$ with $2g < n$:

$$L(\phi, s) = L\left(\chi_{D^*} \otimes f, std, s - \left(\frac{n}{2} - 1\right)\right) \prod_{i=g-\left(\frac{n}{2}-1\right)}^{\left(\frac{n}{2}-1\right)-g} \zeta\left(s + i - \left(\frac{n}{2} - 1\right)\right).$$

If $g = 1$, then obtain $L(\chi_D \otimes \mathrm{Sym}^2(f), s)$ and zeta factors so

$$\lambda_{p,1} = a_p^2 - \chi_{D^*}(p) p^{\frac{n}{2}-1} + p\left(\frac{p^{n-3} - 1}{p - 1}\right)$$

where a_p are the eigenvalues of f.

Example - Nonsquare discriminant

Let Λ be as before with discriminant 29. By checking isometry we compute w.r.t. basis $[\Lambda'], [\Lambda]$

$$[T_2] = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}, [T_3] = \begin{pmatrix} 4 & 3 \\ 6 & 7 \end{pmatrix}, [T_5] = \begin{pmatrix} 18 & 9 \\ 18 & 27 \end{pmatrix}, \ldots$$

The constant function $\phi_1 = [\Lambda] + [\Lambda']$ is an **Eisenstein series** with $T_p(\phi_1) = (p^2 + (1 + \chi_{29}(p)) + 1)\phi_1$. Another eigenvector is $\phi_2 = [\Lambda] - 2[\Lambda']$, with $T_p(\phi_2) = \lambda_p \phi_2$

$$\lambda_2 = -1, \lambda_3 = 1, \lambda_5 = 9, \lambda_7 = 4, \lambda_{11} = 17, \ldots$$

But

$$\theta(\phi_2) = q - \frac{3}{2}q^2 + \frac{3}{2}q^3 - 3q^4 - 3q^5 + O(q^6)$$

is not an eigenform. We match it with the **Hilbert modu** labeled 2.2.29.1-1.1-a in the LMFDB.

Towards a bijection?

Would like to have a bijection between **orthogonal modular forms** and **Hilbert modular forms**, but... Consider $Q(x) = x_1^2 + x_2^2 + x_3^2 + x_1x_4 + x_2x_4 + 3x_4^2$ with Gram matrix

$$[T_\Lambda] = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 0 & 6 \end{pmatrix}$$

and $\mathrm{disc}(\Lambda) = 40$.

- Then $\dim S(\Lambda) = 1 \neq 2 = \dim S_2(\mathbb{Z}[\sqrt{10}])$.
- This is because of the lattice Λ_2 with form $Q_2(x) = x_1^2 + x_2^2 + 2x^3 + x_2x_4 + 2x_3x_4 + 2x_4^2$.
- Although $\Lambda_2 \notin \mathrm{gen}(\Lambda_1)$, it is everywhere locally **similar** to Λ_1.

Similarity classes

We define the general orthogonal group

$$\mathrm{GO}(V) = \{g \in \mathrm{GL}(V) : Q(gv) = \mu(g)Q(v), \quad \mu(g) \in F^\times\}$$
$$\mathrm{GO}(\Lambda) = \{g \in \mathrm{GO}(V) : g\Lambda = \Lambda\}$$

and write $\mathrm{GSO}(V)$ and $\mathrm{GSO}(\Lambda)$ for those with $\det(g) > 0$.
Lattices Λ, Π are **similar**, written $\Pi \sim \Lambda$, if there exists $g \in \mathrm{GO}(V)$ such that $g\Lambda = \Pi$.
The **similarity genus** of Λ is

$$\mathrm{sgen}(\Lambda) := \{\Pi \subseteq V : \Lambda_p \sim \Pi_p \text{ for all } p\}.$$

The **similarity class set** $\mathrm{scls}(\Lambda) = \mathrm{sgen}(\Lambda)/\sim$ is the set of (global) similarity classes in $\mathrm{sgen}(\Lambda)$.

GO modular forms

The space of algebraic modular forms for $\mathrm{GO}(V)$ of level Λ (with trivial weight) is

$$M(\mathrm{GO}(\Lambda)) := \{f : \mathrm{scls}(\Lambda) \to \mathbb{Q}\} \simeq \mathbb{Q}^{h_s(\Lambda)}$$

$M(\mathrm{GO}(\Lambda))$ has additional Hecke operators $T_{\mathfrak{p}}$ at split primes.
We say that integral lattices $\Pi \subseteq \Lambda \subseteq V$ are **$\mathfrak{p}$-neighbors** if

$$\Lambda/\Pi \simeq (R/\mathfrak{p}R)^2 \simeq \Pi/\mathfrak{p}\Lambda,$$

and write $N(\Lambda, \mathfrak{p})$ for the set of $\mathfrak{p}$-neighbors of Λ.
For $\mathfrak{p} \nmid \mathrm{disc}(\Lambda)$ define the **Hecke operator**

$$\begin{aligned} T_{\mathfrak{p}} : M(\mathrm{GO}(\Lambda)) &\to M(\mathrm{GO}(\Lambda)) \\ \phi &\mapsto \left([\Lambda'] \mapsto \sum_{\Pi' \in N(\Lambda', \mathfrak{p})} \phi([\Pi']) \right) \end{aligned}$$

Residually binary lattices

We say that Λ is **residually binary at $\mathfrak{p}$** if $\mathrm{rank}(\Lambda/\mathfrak{p}\Lambda) \geq 2$.

Example

The lattice $\mathbb{Z}^4$ with the form $Q(x) = x_1^2 + 7x_2^2 + 7x_3^2 + 49x_4^2$ is not residually binary at 7.

If Λ is **residually binary everywhere**, can write $\Lambda_{\mathfrak{p}} = \Lambda_{\mathfrak{p},1} \perp \Lambda_{\mathfrak{p},2}$ where $\Lambda_{\mathfrak{p},1}$ and $\Lambda_{\mathfrak{p},2}$ are binary, and $\operatorname{disc}\Lambda_{\mathfrak{p},1} = R_{\mathfrak{p}}$ for every $\mathfrak{p}$. We define the **fundamental discriminant** of Λ to be the ideal $\mathfrak{D} = \mathfrak{D}(\Lambda) \subseteq R$ such that $\operatorname{disc}(\Lambda_{\mathfrak{p},2}) = \mathfrak{D}_{\mathfrak{p}} Q(\Lambda_{\mathfrak{p},2})^2$.

Example

If Λ is maximal, and $K = F[\sqrt{D}]$, then $\mathfrak{D}(\Lambda) = \operatorname{disc} K$.

Let $\mathfrak{M} = \mathfrak{M}(\Lambda)$ be the product of anisotropic primes.

Narrow class number one

In the case where $\mathrm{Cl}^+(F) = 1$, the result is simpler to describe.

Theorem (A., Fretwell, Ingalls, Logan, Secord, and Voight (2022))

Let $\mathrm{disc}(\Lambda) = \mathfrak{D}\mathfrak{N}^2$ *with* $\mathfrak{N}$ *squarefree,* $K = F[\sqrt{D}]$*. Then*

$$S(GO(\Lambda)) \hookrightarrow G_{K|F} \backslash S_2(\mathfrak{N}\mathbb{Z}_K)$$

with image the orbits in $S_2(\mathfrak{N}\mathbb{Z}_K; W = \epsilon)^{\mathfrak{M}\text{-new}}$

- $G_{K|F} = \mathrm{Gal}(K|F)$ acts naturally on the space of Hilbert modular forms.
- For $\mathfrak{p} \mid \mathfrak{N}$, we set $\epsilon_{\mathfrak{p}} = -1$ if $\mathfrak{p} \mid \mathfrak{M}$, else $\epsilon_{\mathfrak{p}} = 1$.
- $W_{\mathfrak{p}}$ is the Atkin-Lehner involution at $\mathfrak{p}\mathbb{Z}_K \mid \mathfrak{N}\mathbb{Z}_K$.

The other forms

- The space of orthogonal modular forms of **weight** (k, j) is

$$M_{k,j}(\mathrm{GO}(\Lambda)) = \{f : \mathrm{scls}(\Lambda) \to W_{k,j} : f(gx) = \rho_{k,j}(g)f(x)\}.$$

- Twisting by the spinor norm, we obtain all the spaces

$$S_{k_1,k_2}(\mathfrak{N}\mathbb{Z}_K; W = \epsilon)^{\mathfrak{M}\text{-new}}$$

- The space $S(\mathrm{O}(\Lambda))$ is identified as the forms invariant under twists by Hecke characters.
- If disc $V = 1$, $K = F \times F$, so that

$$M_{k_1,k_2}(\mathfrak{N}\mathbb{Z}_K) = M_{k_1}(\mathfrak{N}) \otimes M_{k_2}(\mathfrak{N}).$$

When $F = \mathbb{Q}$, this case was considered by Böcherer and Schulze-Pillot (1991).

Special groups and Galois action

Can also define $M(\mathrm{SO}(\Lambda))$ and $M(\mathrm{GSO}(\Lambda))$. If $\mathfrak{p}$ is split, $\mathfrak{p}\mathbb{Z}_K = \mathfrak{P}_1\mathfrak{P}_2$, then

$$T_{\mathfrak{p}} = T_{\mathfrak{P}_1} + T_{\mathfrak{P}_2}, \quad T_{\mathfrak{p},2} = T_{\mathfrak{P}_1,2} + T_{\mathfrak{P}_2,2},$$

coming from splitting of the $\mathfrak{p}^2$-neighbors ($\mathfrak{p}$-neighborhoods) to two orbits.
Since over a local field, every lattice is stable under a reflection, the natural quotient map

$$M(\mathrm{GSO}(\Lambda)) \to M(\mathrm{GO}(\Lambda))$$

induces an isomorphism

$$M(\mathrm{GO}(\Lambda)) = M(\mathrm{GSO}(\Lambda))_{\mathrm{GO}(V)/\,\mathrm{GSO}(V)},$$

and $\mathrm{GO}(V)/\,\mathrm{GSO}(V) \simeq G(K|F)$.

Key ideas - Quaternions and even Clifford

The even Clifford algebra $B = C_0(V)$ is quaternion with center K. Even Clifford extends to a functor

$$C_0 : \mathrm{GSO}(V) \to (B^\times \times F^\times)/K^\times.$$

Theorem (A., Fretwell, Ingalls, Logan, Secord, and Voight (2024))

The even Clifford functor induces an isomorphism

$$C_0^* : M_\rho(C_0(\Lambda)^\times, \psi^{-1} \circ \mathrm{Nm}_{K|F})^{AL_F(C_0(\Lambda))} \longrightarrow M_{C_0^*\rho}(\mathrm{GSO}(\Lambda), \psi).$$

- Sends $\mathfrak{P}$-neighbors to $\mathfrak{P}$-neighbors.
- Sends $\mathfrak{p}^1$-neighbors to $\mathfrak{p}\mathbb{Z}_K$-neighbors.
- Also induces $C_0 : \mathrm{GO}(V)/F^\times \to \mathrm{Aut}_F(B)$, with

$$0 \to B^\times/K^\times \simeq \mathrm{Aut}_K(B) \to \mathrm{Aut}_F(B) \to \mathrm{Gal}(K|F) \to 0.$$

• • $\cong$: [$A_1 \times A_1 = D_2$, equiv. $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \cong \mathfrak{so}_4$]

General narrow class number

If $\mathrm{Cl}^+(F) \neq 1$, $M(\mathrm{GO}(\Lambda)) \longrightarrow M(\mathrm{O}(\Lambda))$ is no longer surjective!

Example

Let $F = \mathbb{Q}(\sqrt{3})$, and consider the lattice

$$[\Lambda] = \begin{pmatrix} 2 & 0 & 0 & \sqrt{3} \\ 0 & 50 & 15\sqrt{3} & 10\sqrt{3} \\ 0 & 15\sqrt{3} & 14 & 9 \\ \sqrt{3} & 10\sqrt{3} & 9 & 8 \end{pmatrix}$$

with $\operatorname{disc}\Lambda = 25\mathbb{Z}_F$, and consider a lattice Λ' with $[\Lambda'] = \varepsilon[\Lambda]$, where $\varepsilon = 2 + \sqrt{3} \in R_{>0}^{\times}$.
Then $\Lambda \sim \Lambda'$, and $\Lambda_{\mathfrak{p}} \simeq \Lambda'_{\mathfrak{p}}$ for all $\mathfrak{p}$ but $\Lambda \not\simeq \Lambda'$.
Thus the natural map $\mathrm{cls}(\Lambda) \to \mathrm{scls}(\Lambda)$ is not injective.

Solution and minusforms

Proposition

There are certain finite abelian groups $U = R^{\times}_{>0}/R^{\times 2}$ *and* $X_\mu = \mathrm{Cl}_{>_F 0}(K)^\vee$ *such that for every character* ψ_U *on* U*, we have*

$$M(\mathrm{SO}(\Lambda), \psi_U) \simeq M(\mathrm{GSO}(\Lambda), \psi_U^{-1} \circ \mu)^{X_\mu}$$

When ψ_U is nontrivial, C_0^* maps these to Hilbert modular forms for $\mathrm{SL}_2(\mathbb{Z}_K)$ with unit character ψ_U.

Example

Let $F = \mathbb{Q}(\sqrt{3})$ and Λ with $\operatorname{disc}\Lambda = 25\mathbb{Z}_F$ as above, and let $\psi_U(\varepsilon) = -1$ be nontrivial. We obtain an eigenform ϕ with

$$\lambda_7 = 64, a_{\mathfrak{p}_{11}} = 24, \lambda_{\mathfrak{p}_{13}} = 4, \lambda_{17} = 4, \lambda_{19} = 196,$$

which corresponds to $f \in S_2(\Gamma_0^1(5\mathbb{Z}_F))$ for $\mathrm{SL}_2(\mathbb{Z}_F)$, with

$$a_7 = 8, a_{\mathfrak{p}_{11}} = \sqrt{24}, a_{\mathfrak{p}_{13}} = 2, a_{17} = 2, a_{19} = 14.$$

Applications - non vanishing

We obtain commutative diagrams of Hecke modules

$$\begin{array}{ccc} S(\mathcal{O})_{G_K}^{AL_F(\mathcal{O})} & \overset{C_0^*}{\longleftrightarrow} & S(\mathrm{GO}(\Lambda)) \\ \Big\updownarrow{\scriptstyle JL} & & \Big\downarrow{\scriptstyle \theta_2} \\ S(\mathfrak{N}\mathbb{Z}_K, W=\epsilon)_{G_K}^{\mathfrak{M}\text{-new}} & \longrightarrow & S^{(2)}(\Gamma_0^{(2)}(\mathfrak{N}), \chi_K) \end{array}$$

The bottom line is:

- Yoshida lift when $K = F \times F$ and f, g are both cuspidal.
- Saito-Kurokawa lift when $K = F \times F$ otherwise.
- Asai lift when K is a field.

Shows that when K is a field (e.g. $D = 4p$ (Kaylor, 2019)), θ_2 is injective (non-vanishing).

Applications - Hilbert modular surfaces

Corollary ((Chan, 1999), Theorem 1)

Let $K = \mathbb{Q}(\sqrt{D})$, and let Y_ϵ be the components of the Hilbert modular surface over K.

$$\sum_{\epsilon \in \mathrm{Cl}^+(K)/\mathrm{Cl}^+(K)^2} (p_g(Y_\epsilon) + 1) = \sum_{\Lambda} \# \operatorname{cls}(\mathrm{SO}(\Lambda)),$$

where Λ ranges over genera of lattices of discriminant D.

Proof.

Main theorem implies $\dim_{\mathbb{Q}} S_2(\mathbb{Z}_K) = \dim_{\mathbb{Q}} S(\mathrm{GSO}(\Lambda))$. But

$$\# \mathrm{Cl}^+(K)/\mathrm{Cl}^+(K)^2 = \#\mu(\operatorname{cls}(\mathrm{GSO}(\Lambda))) = \dim_{\mathbb{Q}} E(\mathrm{GSO}(\Lambda)),$$

$$\dim_{\mathbb{Q}} S_2(\mathbb{Z}_K) + \# \mathrm{Cl}^+(K)/\mathrm{Cl}^+(K)^2 = \dim_{\mathbb{Q}} M(\mathrm{GSO}(\Lambda)),$$

which yields the result. □

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